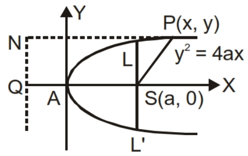
PARABOLA

DEFINITION

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed straight line (called the directrix).



Let S be the focus. QN be the directrix and P be any point on the parabola. Then by definition. PS = PN where PN is the length of the perpendicular from P on the directrix QN.

TERMS RELATED TO PARABOLA

Axis : A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.

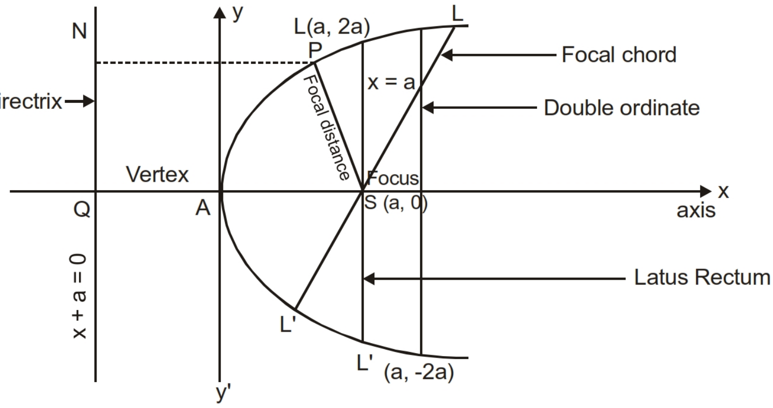
Vertex : The point of intersection of a parabola and its axis is called the vertex of the parabola.

The vertex is the middle point of the focus and the point of intersection of axis and directrix.

Eccentricity : If P be a point on the parabola and PN and PS are the distance from the directrix and focus S respectively then the ratio PS/PN is called the eccentricity of the parabola which is denoted by e.

By the definition for the parabola e = 1.

If $e > 1 \Rightarrow$ hyperbola, $e = 0 \Rightarrow$ circle, $e < 1 \Rightarrow$ ellipse



Latus Rectum

Let the given parabola be $y^2 = 4ax$. In the figure LSL' (a line through focus ⊥ to axis) is the latus rectum.

Also by definition,

$$LSL' = 2\,(\sqrt{4a.a}) = 4a$$

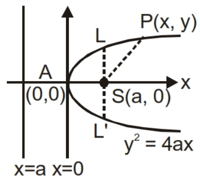
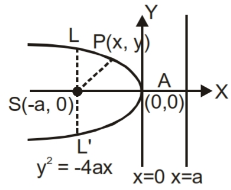
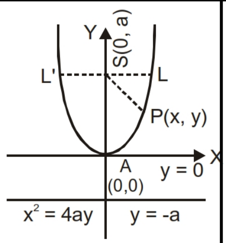
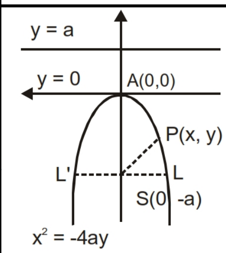
= double ordinate (Any chord of the parabola $y^2 = 4ax$ which is ⊥ to its axis is called the double ordinate) through the focus S.

Note : Two parabolas are said to be equal when their latus recta are equal.

Focal Chord

Any chord to the parabola which passes through the focus is called a focal chord of the parabola.

FOUR STANDARD FORMS OF THE PARABOLA

Standard Equation	$y^2 = 4ax$ (a>0)	$y^2 = -4ax$(a>0)	$x^2 = 4ay$(a>0)	$x^2 = -4ay$(a > 0)
Shape of Parabola				
Vertex	A(0, 0)	A(0, 0)	A(0, 0)	A(0, 0)
Focus	S(a, 0)	S(–a, 0)	S(0, a)	S(0, –a)
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	4a	4a	4a	4a
Extermities of latus rectum	(a, ±2a)	(–a, ±2a)	(±2a, a)	(±2a, –a)
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Equation of tangents at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Focal distance of a point P(x, y)	$x + a$	$x - a$	$y + a$	$y - a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

REDUCTION OF STANDARD EQUATION

If the equation of a parabola contains second degree term either in y or in x(but not in both) then it can be reduced into standard form. For this we change the given equation into the following forms-

$$(y - k)^2 = 4a\,(x - h) \text{ or } (x - p)^2 = 4b\,(y - q)$$

Then we compare from the following table for the results related to parabola.

GENERAL EQUATION OF A PARABOLA

If (h, k) be the locus of a parabola and the euqation of directrix is $ax + by + c = 0$, then its equation is given by

$$\sqrt{(x-h)^2 + (y-k)^2} = \frac{ax + by + c}{\sqrt{a^2 + b^2}} \text{ which gives } (bx - ay)^2 + 2gx + 2fy + d = 0$$

where g, f, d are the constant.

Note

· The general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a parabola, if

(a) $h^2 = ab$

(b) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

PARAMETRIC EQUATIONS OF A PARABOLA

Clearly $x = at^2$, $y = 2at$ satisfy the equation $y^2 = 4ax$ for all real values of t. Hence the parametric equation of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$, where t is the parameter.

Also, $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ for all real values of t. This point is also described as the point 't' on the parabola.

EQUATION OF A CHORD

(i) The equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1y_2$$

(ii) The equation of chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is-

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$y(t_1 + t_2) = 2(x + at_1t_2)$$

(iii) Length of the chord $y = mx + c$ to the parabola $y^2 = 4ax$ is given by $\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$.

Condition for the Chord to be a Focal Chord

The chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus provided $t_1t_2 = -1$.

Length of Focal Chord

The length of a focal chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $(t_2 - t_1)^2$.

Note :

- The length of the focal chord through the point 't' on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$
- The length of the chord joining two points 't_1' and 't_2' on the parabola $y^2 = 4ax$ is

$$a(t_1 - t_2)\sqrt{(t_1 + t_2)^2 + 4}$$

CONDITION FOR TANGENCY AND POINT OF CONTACT

The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Note

- The line $y = mx + c$ touches parabola $x^2 = 4ay$ if $c = -am^2$
- The line $x \cos\alpha + y \sin\alpha = p$ touches the parabola $y^2 = 4ax$ if $a\sin^2\alpha + p\cos\alpha = 0$.

EQUATION OF TANGENT IN DIFFERENT FORMS

(i) Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

Note :

(ii) Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$ty = x + at^2.$$

(iii) Slope Form

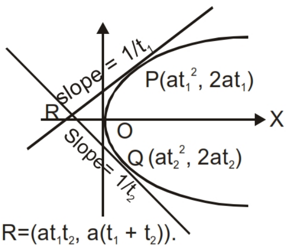
The equation of tangent to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx + \frac{a}{m}.$$

The coordinate of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



$R = (at_1t_2, a(t_1 + t_2))$.

Note :

- Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y_2 = 4ax$ is $\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$
- The G.M. of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The A.M. of the y-coodinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

EQUATIONS OF NORMAL IN DIFFERENT FORMS

(i) Point form

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(ii) Parametric form

The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$

(iii) Slope form

The eqaution of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx - 2am - am^3$$

Note : The coordinates of the point of contact are $(am^2 - 2am)$.

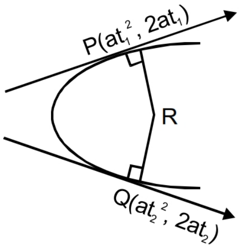
Condition for Normality

The line $y = mx + c$ is normal to the parabola

$$y^2 = 4ax \quad \text{if } c = -2am - am^3 \quad \text{and} \quad x^2 = 4ay \quad \text{if } c = 2a + \frac{a}{m^2}$$

Point of Intersection of Normals

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

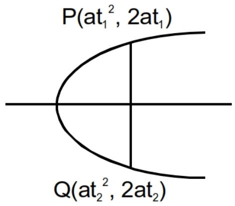


$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2),\ -at_1t_2(t_1 + t_2)]$$

Note :

- If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$



It is clear that PQ is normal to the parabola at P and not at Q.

- If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$
- The normal at the extremities of the latus rectum of a parabola intersect at right angle on the axis of the parabola.

Co-normal Points

Any three points on a parabola normals at which pass through a common point are called co-normal points

Note :

This implies that if three normals are drawn through a point (x_1, y_1) then their slopes are the roots of the cubic: $y_1 = mx_1 - 2am - am^3$ which gives three values of m. Let these values are m_1, m_2, m_3 then from the eq^n.

$\Rightarrow \qquad am^3 + (2a - x_1)m + y_1 = 0$

- The sum of the slopes of the normals at co-normal points is zero, i.e., $m_1 + m_2 + m_3 = 0$.

  and $m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a - x_1}{a}$ and $m_1m_2m_3 = -\dfrac{y_1}{a}$

- The sum of the ordinates of the co-normal points is zero (i.e., $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$.
- The centroid of the triangle formed by the co-normal points lies on the axis of the parabola

- The vertices of the triangle formed by the co-normal points are $(am_1^2 - 2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, y-coordinate of the centroid becomes

$$\frac{-2a(m_1+m_2+m_3)}{3} = \frac{-2a}{3}\times 0 = 0.$$

i.e., centroid of triangle $\left(\frac{am_1^2+am_2^2+am_3^2}{3}, \frac{2am_1+2am_2+2am_3}{3}\right) = \left(\frac{am_1^2+am_2^2+am_3^2}{3}, 0\right)$

Hence, the centroid lies on the x-axis i.e., axis of the parabola.]

- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.

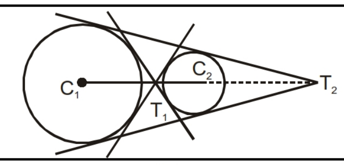
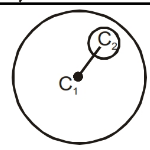
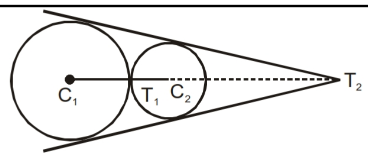
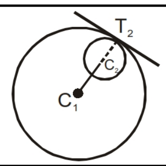
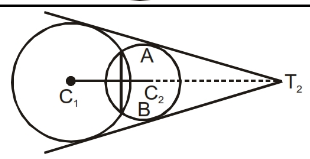
POSITION OF A POINT & LINE W.R.T. A PARABOLA

- The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, = \text{ or } < 0$, respectively.
- The line $y = mx + c$ will intersect a parabola $y^2 = 4ax$ in two real and different, coincident or imaginary point, according as $a - mc >, = < 0$.

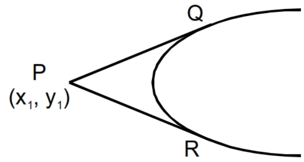
Number of tangents drawn from a point to a parabola

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

EQUATION OF THE PAIR OF TANGENTS

	CONDITION	POSITION	DIAGRAM	NO. OF COMMON TANGENTS
(i)	$C_1C_2 > r_1 + r_2$	do not intersect or one outside the other		4
(ii)	$C_1C_2 < \lvert r_1 - r_2 \rvert$	one inside the other		0
(iii)	$C_1C_2 = r_1 + r_2$	external touch		3
(iv)	$C_1C_2 = \lvert r_1 - r_2 \rvert$	internal touch		1
(v)	$\lvert r_1 - r_2 \rvert < C_1C_2 < r_1 + r_2$	intersection at two real points		2

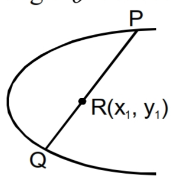
The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$.



where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1{}^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$

LOCUS OF POINT OF INTERSECTION

The locus of point of intersection of tangent to the parabola $y^2 = 4ax$ which are having an angle θ between them given by $y^2 - 4ax = (a + x)^2 \tan^2 \theta$

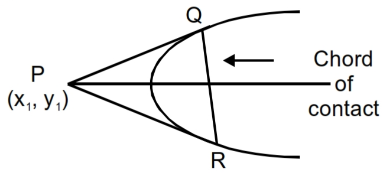


Note :

- If $\theta = 0°$ or p then locus is $(y^2 - 4ax) = 0$ which is the given parabola.
- If $\theta = 90°$, then locus is $x + a = 0$ which is the directrix of the parabola.

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.



Note :

- The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- Lengths of the chord of contact is $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$
- Area of triangle formed by tangents drawn from (x_1, y_1) and their chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$.

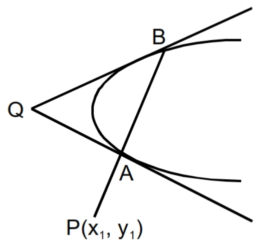
CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1 \quad \text{where } T \equiv yy_1 - 2a(x + x_1) \text{ and } S_1 \equiv y_1{}^2 - 4ax.$$

POLE AND POLAR

Let P be a given point. Let a line through P intersect the parabola at two points A and B. Let the tangents at A anf B intersect at Q. The locus of point Q is a straight line called the polar of point P with respect to the parabola and the point P is called the pole of the polar.



Equation of Polar of a Point

The polar of a point $P(x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

Coordinate of pole

The pole of the line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{\ell}, -\frac{2am}{\ell}\right)$

Conjugate points and conjugate lines

(i) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

So two points (x_1, y_1) and (x_2, y_2) are conjugate points with respect to parabola $y^2 = 4ax$ if $yy_1 = 2a(x_1+x_2)$

(ii) If the pole of a line $ax + by + c = 0$ lies on the another line $a_1x + b_1y + c_1 = 0$ then the pole of the second line will lie on the first and such line are said to be conjugate lines.

So two lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate lines with respect to parabola $y_2 = 4ax$ if $(l_1n_2 + l_2n_1) = 2\,am_1m_2$

Note

· The polar of focus is directrix and pole of directrix is focus.

· The polars of all points on directrix always pass through a fixed point and this fixed point is focus.

· The pole of a focal chord lies on directrix and locus of poles of focal chord is a directrix.

· The chord of contact and polar of any point on the directrix always passes through focus.

DIAMETER OF A PARABOLA

Diameter of a parabola is the locus of middle points of a series of its parallel chords.

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.